Arch. Math. 83 (2004) 70–80
0003–889X/04/010070–11
DOI 10.1007/s00013-004-4760-2

Archiv der Mathematik

Long line knots

By

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Abstract. We study continuous embeddings of the long line L into L^n ($n \geqq 2$) up to ambient isotopy of L^n. We define the direction of an embedding and show that it is (almost) a complete invariant in the case $n = 2$ for continuous embeddings, and in the case $n \geqq 4$ for differentiable ones. Finally, we prove that the classification of smooth embeddings $\mathsf{L} \to \mathsf{L}^3$ is equivalent to the classification of classical oriented knots.

Introduction. Consider the following general problem:

$$(\star) \qquad \begin{array}{c}\text{Given finite dimensional manifolds } X, Y, \text{ classify} \\ \text{the embeddings } X \to Y \text{ up to ambient isotopy of } Y.\end{array}$$

An instance of this problem is classical knot theory, where $X = \mathbb{S}^1$, $Y = \mathbb{S}^3$ and the embeddings are assumed smooth (or PL). Another instance is higher dimensional knot theory, where $X = \mathbb{S}^k$ and $Y = \mathbb{S}^n$. These fields have been very popular among mathematicians for over a century; there is a considerable literature about the classification of (topological, differentiable or PL) embeddings from a k-sphere to an n-sphere.

On the other hand, the study of this problem for non-metrizable manifolds remains to be done. Even the study of homotopy and isotopy classes of maps of non-metrizable manifolds seems to be at its very beginning. To our knowledge, David Gauld [2] was the first ever to publish a paper on the subject (see also [1]). He investigated homotopy classes of maps of the *long line* L (and of the *long ray* R) into itself (see Definition 1.1 below), showing that there are exactly 2 such classes for the long ray, and 9 for the long line. He also proved that all embeddings $\mathsf{R} \to \mathsf{R}$ are isotopic, and that there are 2 isotopy classes of embeddings $\mathsf{L} \to \mathsf{L}$. These results provide a solution to Problem ($\star$) for $X = Y = \mathsf{R}, \mathsf{L}$.

In this paper, we investigate Problem ($\star$) with $X = \mathsf{L}$, $Y = \mathsf{L}^n$ and $X = \mathsf{R}$, $Y = \mathsf{R}^n$. We introduce a numerical invariant of embeddings $\mathsf{R} \to \mathsf{R}^n$ or $\mathsf{L} \to \mathsf{L}^n$ called the *direction*, see Definition 3.1 and Theorem 3.2. Roughly speaking, two knots (that is, two embeddings) have the same direction if for all $1 \leqq i \leqq n$, their projections on the i-th coordinate are either both cofinal or both bounded.

Mathematics Subject Classification 2000: 57N37, 03E10, 57M25.

Our first results are that the direction is (almost) a complete invariant in the cases $n = 2$ for continuous embeddings (Theorems 3.3 and 3.5) and $n \geqq 4$ for smooth embeddings (Theorem 3.6 and 3.7). It follows that for these n, there are exactly $2^n - 1$ long ray knots and $(3^n - 1)^2 + (n-2)2^{n-1}$ long line knots (in the classes specified above). We also prove that there are 7 smooth knots $\mathsf{R} \to \mathsf{R}^3$ (Theorem 3.8). Finally, our last result shows that the classification of differentiable embeddings $\mathsf{L} \to \mathsf{L}^3$ reduces to classical oriented knot theory (see Proposition 3.12 and Theorem 3.13).

The paper is organized as follows: Section 1 deals with the definition of some basic objects, such as the long ray R, the long line L, and the equivalence relation for knots. In Section 2, we prove several technical lemmas (mainly partition and covering properties) that are used in Section 3, which contains all the main results.

The authors wish to acknowledge Claude Weber, David Gauld, and René Binamé.

1. Definitions. For any ordinal α, let us denote by $W(\alpha)$ the set of ordinals strictly smaller than α.

Definition 1.1. The (closed) long ray R is the set $W(\omega_1) \times [0, 1[$, equipped with the lexicographic order and the order topology. The long line L is the union of two copies L_-, L_+ of R glued at $(0, 0)$. We put the reverse order on L_-, so that L is totally ordered.

We will often identify $\alpha \in W(\omega_1)$ with $(\alpha, 0) \in \mathsf{R}$ and similarly for L. Recall that R and L are non-metrizable, non-contractible and sequentially compact. Also, L and R can be given a structure of oriented $\mathcal{C}^\infty$ manifold. We will assume throughout the text that we are given a *fixed* maximal atlas $\{U_j, \psi_j\}$ on L (and on R) with $U_j \ni 0$ for all j. The atlas on L^n (or R^n) is then assumed to be $\{(U_j)^n, (\psi_j, \dots, \psi_j)\}$, the so-called '$n$-th power structure'.[1] This assumption is, in our opinion, quite natural. For instance, this ensures that the maps $\mathsf{R} \to \mathsf{R}^2$, given by $x \mapsto (x, x)$ and $x \mapsto (x, c)$ are smooth, where c is any constant. Throughout the text, π_i will denote the projection on the i-th coordinate of L^n or R^n.

Definition 1.2. We call an embedding from X to Y a (Y, X)-knot. Two $(\mathsf{L}^n, \mathsf{L})$-knots f, g are equivalent, which we denote by $f \sim g$, if there is an isotopy ϕ_t $(t \in [0, 1])$ of L^n such that $\phi_0 = \mathrm{id}$ and $\phi_1 \circ f = g$.

Since R is a manifold with boundary, we have to be more careful with the definition of equivalent $(\mathsf{R}^n, \mathsf{R})$-knots. One possibility is to consider only embeddings of R in the interior of R^n. Here is another way to state the same equivalence relation:

Definition 1.3. Let R' be the set $(\{-1\} \times [0, 1[) \sqcup \mathsf{R}$ equipped with the order topology. Two $(\mathsf{R}^n, \mathsf{R})$-knots f, g are equivalent, which we denote by $f \sim g$, if there is an isotopy ϕ_t $(t \in [0, 1])$ of $(\mathsf{R}')^n$ such that $\phi_0 = \mathrm{id}$ and $\phi_1 \circ f = g$.

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[1] These precisions are necessary, since P.J. Nyikos showed [4] that there are uncountably many non-equivalent differential structures on L (and thus, on L^n). Moreover, it is not clear that any differential structure on L^n is equivalent to a product of structures on L.

2. Tools. Let us begin by recalling the following well-known lemma; the proof of the corresponding statement for ordinals can be found in any book on set theory.

Lemma 2.1. *Let* $\{E_m\}_{m<\omega}$ *be closed and cofinal subsets of* R. *Then* $\bigcap_{m<\omega} E_m$ *is also closed and cofinal in* R.

We shall now investigate partition and covering properties of embeddings $\mathsf{R} \to \mathsf{R}^n$.

Lemma 2.2. *Let* $\{f_k\}_{k\in K}$ *be a finite or countable family of continuous maps* $\mathsf{R} \to \mathsf{R}$.

a) *If each* f_k *is bounded, there is a* z *in* R *such that* f_k *is constant on* $[z, \omega_1[$ *for all* $k \in K$.
b) *If each* f_k *is cofinal, there is a cover* $\mathcal{P} = \{[x_\alpha, x_{\alpha+1}]\}_{\alpha<\omega_1}$ *of* R *such that* $x_\alpha < x_\beta$ *if* $\alpha < \beta$, $x_\beta = \sup_{\alpha<\beta} x_\alpha$ *if* β *is a limit ordinal, and for all* $k \in K$, $f_k([x_\alpha, x_{\alpha+1}]) = [x_\alpha, x_{\alpha+1}]$ *for* $\alpha > 0$ *and* $f_k([0, x_1]) \subset [0, x_1]$.

Proof. a) By [3, Lemma 3.4 (iii)], there exists $z_k \in \mathsf{R}$ such that f_k is constant on $[z_k, \omega_1[$. Take $z = \sup_{k\in K} z_k$.
b) This is a consequence of the following claims:

Claim A. $A_k \overset{\text{def}}{=} \{x \in \mathsf{R} \mid f_k(y) \leqq x \ \forall y \leqq x\}$ is closed and cofinal in R.

Claim B. $B_k \overset{\text{def}}{=} \{x \in \mathsf{R} \mid f_k(y) \geqq x \ \forall y \geqq x\}$ is closed and cofinal in R.

Indeed, these two claims together with Lemma 2.1 imply that the set $E = \bigcap_{k\in K} (A_k \cap B_k) = \{x \in \mathsf{R} \mid f_k([0, x]) \subset [0, x] \text{ and } f_k([x, \omega_1[) \subset [x, \omega_1[\ \forall k \in K\}$ is closed and cofinal. We then define by transfinite induction $x_\alpha \in E$ for each $\alpha < \omega_1$ as follows. Set $x_0 = 0$, $x_{\alpha+1} = \min(E \cap [x_\alpha + 1, \omega_1[)$, and $x_\beta = \sup_{\alpha<\beta} x_\alpha$ if β is a limit ordinal. (Recall that for any limit ordinal $\beta < \omega_1$, there is a sequence $\{\alpha_i\}_{i<\omega}$, with $\alpha_i < \beta$, such that $\lim_{i\to\infty} \alpha_i = \beta$. Therefore, $x_\beta = \lim_{i\to\infty} x_{\alpha_i}$ belongs to E.)

Proof of Claim A. Closeness is obvious (recall that a sequentially closed subset of R is closed). Given z in R, one can choose a sequence $\{x_j\}_{j<\omega}$ such that $z < x_j \leqq x_{j+1}$ and $\sup_{[0,x_j]} f_k \leqq x_{j+1}$. The sequence converges to a point $x > z$, and $x \in A_k$.

Proof of Claim B. Again, closeness is obvious. Fix z in R; for any $x \in \mathsf{R}$, there is a $y \in \mathsf{R}$ with $y \geqq x$ such that $f_k([y, \omega_1[) \subset [x, \omega_1[$. (Otherwise, the set of y such that $f_k(y) < x$ is cofinal and closed; by cofinality of f_k, so is the set of y' such that $f_k(y') > x$. By Lemma 2.1, there would exist v such that $f_k(v) < x$ and $f_k(v) > x$, which is impossible.) We then define sequences x_j, y_j $(j < \omega)$ such that $z < x_j \leqq y_j \leqq x_{j+1}$ and $f_k([y_j, \omega_1[) \subset [x_j, \omega_1[$. The sequences x_j, y_j converge to the same point $x > z$, which belongs to B_k. $\square$

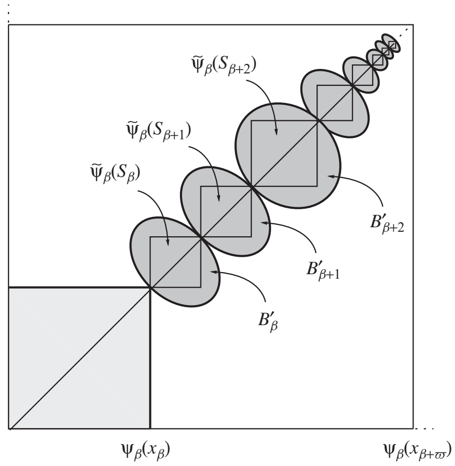

Figure 1. Lemma 2.3 for $n = 2$.

Lemma 2.3. *Let $\mathcal{P} = \{[x_\alpha, x_{\alpha+1}]\}_{\alpha<\omega_1}$ be a cover of R with $x_\alpha < x_\beta$ if $\alpha < \beta$ and $x_\beta = \sup\limits_{\alpha<\beta} x_\alpha$ if β is a limit ordinal, and let S_α be the cube $[x_\alpha, x_{\alpha+1}]^n$. Then, for all $0 < \alpha < \omega_1$, there are closed subsets B_α of R^n, diffeomorphic to a compact ball in $\mathbb{R}^n$, such that*

- $B_\alpha \supset S_\alpha$,
- $\partial B_\alpha \cap S_\alpha = \{(x_\alpha, \dots, x_\alpha), (x_{\alpha+1}, \dots, x_{\alpha+1})\}$,
- $\partial B_\alpha \cap S_\beta = S_\alpha \cap S_\beta$ if $\alpha \neq \beta$.

Proof. Let β be a limit ordinal or 1. Consider $\psi_\beta : U_\beta \to \mathbb{R}$ the smallest chart such that $U_\beta \supset [0, x_{\beta+\omega}]$ (such a chart always exists), and $\widetilde{\psi}_\beta = (\psi_\beta, \dots, \psi_\beta) : (U_\beta)^n \to \mathbb{R}^n$, which belongs to the atlas of R^n. Clearly, $\widetilde{\psi}_\beta(S_\alpha)$ is a cube in $\mathbb{R}^n$ for any $\alpha < \beta + \omega$. We can therefore form ellipsoids B'_α for $\beta \leqq \alpha < \beta + \omega$ as in Figure 1, and define $B_\alpha = \widetilde{\psi}_\beta^{-1}(B'_\alpha)$ for $\beta \leqq \alpha < \beta + \omega$. Note that $B'_\beta \cap \widetilde{\psi}_\beta([0, x_\beta]^n) = \{\widetilde{\psi}_\beta(x_\beta)\}$. Since $W(\omega_1)\backslash\{0\} = [1, \omega[\sqcup \bigsqcup\limits_{\beta \text{lim.}} [\beta, \beta + \omega[$, we obtain the desired properties. $\square$

Lemma 2.4. *Let $f : \mathsf{R} \to \mathsf{R}^n$ be continuous, with $\pi_i \circ f$ bounded for $i = 1, \dots, s$ $(s < n)$ and unbounded for $i = s+1, \dots, n$. Then, there exists a cover $\mathcal{P} = \{[x_\alpha, x_{\alpha+1}]\}_{\alpha<\omega_1}$ of R and a cover $\mathcal{D} = \{D_\alpha\}_{0<\alpha<\omega_1}$ of $f([x_1, \omega_1[)$ with D_α diffeomorphic to the compact ball in $\mathbb{R}^n$, such that (see Figure 2):*

- $f(x) \subset \{c\} \times \mathsf{R}^{n-s}$ $\forall x \geqq x_1$, *for some fixed* $c \in \mathsf{R}^s$,
- $f(x_\alpha) = (c, x_\alpha, \dots, x_\alpha)$ $\forall \alpha > 0$,
- $f^{-1}(D_\alpha) = [x_\alpha, x_{\alpha+1}]$ *and* $f^{-1}(\partial D_\alpha) = \{x_\alpha, x_{\alpha+1}\}$ $\forall \alpha > 0$,
- $D_\alpha \cap D_\beta = \begin{cases} \emptyset & \text{if } \alpha \neq \beta \pm 1 \\ (c, x_\alpha, \dots, x_\alpha) & \text{if } \alpha = \beta + 1. \end{cases}$

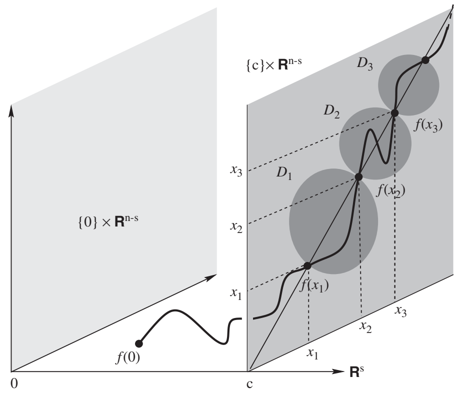

Figure 2. Lemma 2.4 for $n = 3$, $s = 1$.

Proof. By Lemma 2.2 a), there is some $z \in \mathsf{R}$ such that $\pi_i \circ f$ is constant on $[z, \omega_1[$ for $i = 1, \ldots, s$. Set $c_i = \pi_i \circ f(z)$ and $c = (c_1, \ldots, c_s)$. For $i = s + 1, \ldots, n$, take the cover $\mathcal{P}$ for the family $\{\pi_i \circ f\}_{i=s+1,\ldots,n}$ with $x_1 > z$ given by Lemma 2.2 b). By construction, the cover $\mathcal{P} = \{I_\alpha\}_{\alpha<\omega_1}$, with $I_\alpha = [x_\alpha, x_{\alpha+1}]$, satisfies the first two claims. Choose now for each $i = 1, \ldots, s$ a compact interval J_i containing c_i in its interior, and set $J = J_1 \times \ldots \times J_s$. By Lemma 2.3 applied to the cover $\mathcal{P}$, we get closed sets $B_\alpha \subset \mathsf{R}^{n-s}$, such that B_α contains $(I_\alpha)^{n-s}$ for all $\alpha > 0$. Therefore, $f(I_\alpha) \subset D'_\alpha \overset{\text{def}}{=} J \times B_\alpha$ for $\alpha > 0$, and the intersection $\partial D'_\alpha \cap f(I_\alpha)$ is equal to $\{f(x_\alpha), f(x_{\alpha+1})\}$. By construction, $D'_\alpha \cap D'_{\alpha+1} = J \times \{(x_{\alpha+1}, \ldots, x_{\alpha+1})\}$. Since B_α is a $(n-s)$-compact ball and $f(I_\alpha)$ is contained in the hyperplane $x_1 = c_1, \ldots, x_s = c_s$, one can find a subset D_α of D'_α which is diffeomorphic to the compact ball in $\mathbb{R}^n$, that has the desired boundary properties (see Figure 3). $\square$

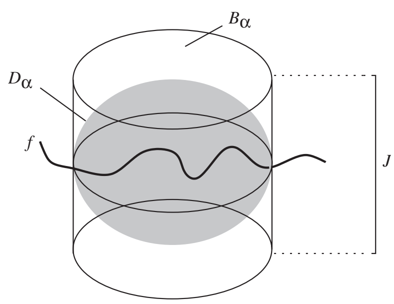

Figure 3. The subset D_α of $J \times B_\alpha$.

Proposition 2.5. *Let B be the closed unit ball in $\mathbb{R}^n$, and let $f, g : [0, 1] \to B$ be continuous embeddings with $f^{-1}(\partial B) = g^{-1}(\partial B) = \{0, 1\}$, $f(0) = g(0)$ and $f(1) = g(1)$.*

Then, there exists an isotopy $\phi_t : B \to B$ $(t \in [0, 1])$ *keeping* ∂B *fixed, such that* $\phi_0 = \mathrm{id}_B$ *and* $\phi_1 \circ f = g$*, if one of the following conditions holds*:

a) $n = 2$,
b) $n \geqq 4$ *and* f, g *are* $\mathcal{C}^1$ *embeddings.*

Suppose now that $f^{-1}(\partial B) = g^{-1}(\partial B) = \{0\}$ *and* $f(0) = g(0)$. *Then, we have the same conclusion if:*

c) $n = 2$,
d) $n \geqq 3$ *and* f, g *are* $\mathcal{C}^1$ *embeddings.*

Proof. Points a) and c) are direct consequences of Schönfliess theorem. Point b) follows from Zeeman's unknotting theorem (see e.g. [5, Theorem 7.1]). Point d) follows from assertion b) for $n \geqq 4$; for $n = 3$, consider a good projection. □

3. Classification of long knots. We are now ready to present our main results.

Definition 3.1. The direction of a continuous function $f : \mathsf{R} \to \mathsf{R}^n$ is the vector $D(f) = (\delta_1(f) \dots \delta_n(f))^T$, where

$$\delta_i(f) = \begin{cases} 1 & \text{if } \pi_i \circ f \text{ is cofinal;} \\ 0 & \text{if } \pi_i \circ f \text{ is bounded.} \end{cases}$$

If $g : \mathsf{R} \to \mathsf{L}^n$ is continuous, let us define $\delta(g) = (\delta_1(g) \dots \delta_n(g))^T$ as follows:

$$\delta_i(g) = \begin{cases} +1 & \text{if } \pi_i \circ g \text{ is cofinal in } \mathsf{L}_+; \\ -1 & \text{if } \pi_i \circ g \text{ is cofinal in } \mathsf{L}_-; \\ \ \ 0 & \text{if } \pi_i \circ g \textit{ is bounded}. \end{cases}$$

The direction of a continuous fonction $f : \mathsf{L} \to \mathsf{L}^n$ is the $(n \times 2)$-matrix $D(f) = (\delta(f|_{\mathsf{L}_-})\delta(f|_{\mathsf{L}_+}))$.

A matrix D is a (Y, X)-direction ($X = \mathsf{R}, \mathsf{L}$, $Y = \mathsf{R}^n, \mathsf{L}^n$) if $D = D(f)$ for some continuous $f : X \to Y$.

Theorem 3.2. *The direction is an invariant of* $(\mathsf{R}^n, \mathsf{R})$ *and* $(\mathsf{L}^n, \mathsf{L})$*-knots.*

Proof. Let $f, g : \mathsf{R} \to \mathsf{R}^n$ be two continuous embeddings with $D(f) \neq D(g)$. Without loss of generality, it may be assumed that $\pi_1 \circ f$ is bounded and $\pi_1 \circ g$ cofinal. Then, f and g are not equivalent. Indeed, consider an isotopy ϕ_t of $(\mathsf{R}')^n$ such that $\phi_0 = \mathrm{id}$ and $\phi_1 \circ f = g$; then, $\pi_1 \circ \phi_t \circ f : \mathsf{R} \to \mathsf{R}'$ provides a homotopy between $\pi_1 \circ f : \mathsf{R} \to \mathsf{R} \subset \mathsf{R}'$ and $\pi_1 \circ g : \mathsf{R} \to \mathsf{R} \subset \mathsf{R}'$. Since $\pi_1 \circ f$ is bounded, it follows from Lemma 2.2 a) that $\pi_1 \circ f$ is homotopic to a constant. On the other hand, Lemma 2.2 b) implies that $\pi_1 \circ g$ is homotopic to the canonical inclusion $\mathsf{R} \subset \mathsf{R}'$. Therefore, we would have a homotopy from a constant map to the inclusion $\mathsf{R} \subset \mathsf{R}'$. Since this inclusion is a homotopy equivalence, and since R is not contractible (see e.g. [2]), such an homotopy does not exist. The proof for $(\mathsf{L}^n, \mathsf{L})$-knots is very similar. □

(R^2, R) and (L^2, L)-knots.

Theorem 3.3. *The direction is a complete invariant for* $(\mathsf{R}^2, \mathsf{R})$*-knots. Therefore, there are exactly* 3 *classes of non-equivalent* $(\mathsf{R}^2, \mathsf{R})$*-knots.*

Proof. There are 4 possible directions for a continuous map $f : \mathsf{R} \to \mathsf{R}^2$. If f is an embedding, the direction $D = \binom{0}{0}$ is impossible. (Otherwise, by Lemma 2.2 a), there would be some $z \in \mathsf{R}$ such that f is constant on $[z, \omega_1[$.) Therefore, we are left with three possible directions, realized by the following $(\mathsf{R}^2, \mathsf{R})$-knots: $f_{\binom{0}{1}}(x) = (0, x)$, $f_{\binom{1}{0}}(x) = (x, 0)$ and $f_{\binom{1}{1}}(x) = (x, x)$. By Theorem 3.2, we just need to show that an $(\mathsf{R}^2, \mathsf{R})$-knot f with direction D is equivalent to f_D, for $D = \binom{1}{1}, \binom{0}{1}$, and $\binom{1}{1}$.

Let us first assume that $D = \binom{1}{1}$, and let $\{I_\alpha = [x_\alpha, x_{\alpha+1}]\}_{\alpha<\omega_1}$ and $\{D_\alpha\}_{0<\alpha<\omega_1}$ be the covers given by Lemma 2.4. Each D_α satisfies the hypotheses of Proposition 2.5 a), with $f = f|_{I_\alpha}$ and $g = f_{\binom{1}{1}}|_{I_\alpha}$. Thus, we can find isotopies $(\phi_\alpha)_t$ of D_α (rel ∂D_α) such that $(\phi_\alpha)_0 = \mathrm{id}$ and $(\phi_\alpha)_1 \circ f|_{I_\alpha} = f_{\binom{1}{1}}|_{I_\alpha}$. Then, consider a set $D_0 \subset (\mathsf{R}')^2$, diffeomorphic to the compact ball, that contains $f([0, x_1[) \cup f_{\binom{1}{1}}([0, x_1[)$ in its interior, and such that $D_0 \cap D_1 = \{(x_1, x_1)\}$. Then, D_0 satisfies the assumptions of Proposition 2.5 c), and there is an ambient isotopy $(\phi_0)_t$ of D_0 (rel ∂D_0) between $f|_{I_0}$ and $f_{\binom{1}{1}}|_{I_0}$. Extending the ambient isotopies $(\phi_\alpha)_t$ $(\alpha < \omega_1)$ by the identity outside $\cup_{\alpha<\omega_1} D_\alpha$, we have proved that $f \sim f_{\binom{1}{1}}$.

Now, consider the case where $D = \binom{1}{0}$ (the case $D = \binom{0}{1}$ is similar). As before, take the cover $\{D_\alpha\}_{0<\alpha<\omega_1}$ of $f([x_1, \omega_1[)$ given by Lemma 2.4, and choose $D_0 \subset (\mathsf{R}')^2$, diffeomorphic to the compact ball, that contains $f([0, x_1[) \cup (\{c_1\} \times [0, x_1[)$ in its interior, and such that $D_0 \cap D_1 = \{(c_1, x_1)\}$. By Proposition 2.5 c), there is an ambient isotopy of D_0 (rel ∂D_0) that sends $f([0, x_1])$ on $\{c_1\} \times [0, x_1]$. Extending it by the identity outside D_0, we have an ambient isotopy between $\mathrm{Im} f$ and $\{c_1\} \times \mathsf{R}$. It is now straightforward to define an isotopy between $\{c_1\} \times \mathsf{R}$ and $\{0\} \times \mathsf{R} = \mathrm{Im} f_{\binom{1}{0}}$. Therefore, we can assume that $\mathrm{Im} f = \mathrm{Im} f_{\binom{1}{0}} = \{(x, 0) \mid x \in \mathsf{R}\}$. In that case, $\pi_1 \circ f$ is an homeomorphism of R. By [2, Corollary 2], there is an isotopy γ_t of R (keeping 0 fixed) such that $\gamma_0 = \mathrm{id}_{\mathsf{R}}$ and $\gamma_1 = \pi_1 \circ f$. Then, the isotopy ϕ_t of $(\mathsf{R}')^2$ given by $\phi_t(x, y) = (\gamma_t(x), y)$ if $x \in \mathsf{R}$ and $\phi_t(x, y) = (x, y)$ otherwise satisfies $\phi_0 = \mathrm{id}$ and $\phi_1 \circ f_D = f$. $\square$

The direction is almost a complete invariant for $(\mathsf{L}^2, \mathsf{L})$-knots. It fails to be complete only because some directions correspond to exactly two non-equivalent knots. This phenomenon also appears in dimension $n \geqq 2$, motivating the following definition.

Definition 3.4. A double direction is an $(\mathsf{L}^n, \mathsf{L})$-direction with equal columns that contain exactly one 0.

Theorem 3.5. *There are exactly* 64 *non-equivalent* $(\mathsf{L}^2, \mathsf{L})$*-knots.*

Proof. Given $f : \mathsf{L} \to \mathsf{L}^2$ continuous, there are 9^2 possible direction matrices $(\delta(f|_{\mathsf{L}_-})$ $\delta(f|_{\mathsf{L}_+}))$. If f is an $(\mathsf{L}^2, \mathsf{L})$-knot, the restrictions $f|_{\mathsf{L}_-}$ and $f|_{\mathsf{L}_+}$ are embeddings; by

Lemma 2.2 a), this implies that the columns $\delta(f|_{\mathsf{L}_-})$ and $\delta(f|_{\mathsf{L}_+})$ are non-zero. Furthermore, the following four directions are also forbidden:

$$\begin{pmatrix} +1 & +1 \\ +1 & +1 \end{pmatrix}, \begin{pmatrix} +1 & +1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ +1 & +1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

(Indeed, let us consider a continuous map $f : \mathsf{L} \to \mathsf{L}^2$ with direction matrix $\binom{+1\,+1}{+1\,+1}$. Since $\delta(f|_{\mathsf{L}_+}) = \binom{+1}{+1}$, it follows from Lemma 2.2 b) that the sets $\{x \in \mathsf{L}_+ \mid \pi_1 \circ f(x) = x\}$ and $\{x \in \mathsf{L}_+ \mid \pi_2 \circ f(x) = x\}$ are closed and cofinal. By Lemma 2.1, so is their intersection $\{x \in \mathsf{L}_+ \mid f(x) = (x, x)\}$. Similarly, $\delta(f|_{\mathsf{L}_-}) = \binom{+1}{+1}$ implies that $\{y \in \mathsf{L}_- \mid f(y) = (-y, -y)\}$ is closed and cofinal. By Lemma 2.1 again, the set $\{z \in \mathsf{L}_+ \mid f(z) = (z, z) = f(-z)\}$ is cofinal, so f is not an embedding. The other three cases are similar.) So, we are left with $(9-1)^2 - 4 = 60$ possible directions for an $(\mathsf{L}^2, \mathsf{L})$-knot. It is easy to exhibit a knot realizing each of these directions.

As in Theorem 3.3, one shows that two knots with the same direction D are equivalent, except if D is a double direction. To each double direction correspond exactly two classes of knots, as we shall see. Since there are four double directions, namely

$$\begin{pmatrix} +1 & +1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ +1 & +1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix},$$

the theorem follows. Let D be the first of these directions, and let $f : \mathsf{L} \to \mathsf{L}^2$ be an embedding with $f(x) = (x, 1)$ for $x \geqq 1$ and $f(x) = (-x, -1)$ for $x \leqq -1$ (as usual, we denote by $-x$ the point in L_- corresponding to $x \in \mathsf{L}_+$, and vice versa). Finally, let $g : \mathsf{L} \to \mathsf{L}^2$ be the knot given by $g(x) = f(-x)$. Clearly, $D = D(f) = D(g)$ and any $(\mathsf{L}^2, \mathsf{L})$-knot with direction D is equivalent to either f or g. It remains to check that f and g are non-equivalent. Indeed, let ϕ_t be an isotopy between f and g, i.e. $\phi_0 = \mathrm{id}$ and $\phi_1 \circ f = g$. Since $\pi_2 \circ \phi_t \circ f$ is a bounded continuous map $\mathsf{L} \to \mathsf{L}$, Lemma 2.2 a) implies that $\pi_2 \circ \phi_t \circ f([x, \omega_1[)$ is a single point for x large enough. Let us denote it by $r_+(t)$, and similarly, let $r_-(t)$ be the element defined by $\pi_2 \circ \phi_t \circ f(]-\omega_1, -x])$ for x large enough. One checks that r_+ and r_- are continuous. Since $r_+(0) = r_-(1) = 1$ and $r_+(1) = r_-(0) = -1$, there is some t_0 for which $r_+(t_0) = r_-(t_0)$. Then, $\phi_{t_0} \circ f$ is not an embedding. The other double directions are similarly treated. $\square$

Differentiable $(\mathsf{R}^n, \mathsf{R})$ and $(\mathsf{L}^n, \mathsf{L})$-knots for $n \geqq 4$.

Theorem 3.6. *There are exactly $(2^n - 1)$ non-equivalent differentiable $(\mathsf{R}^n, \mathsf{R})$-knots if $n \geqq 4$.*

Proof. Given f an $(\mathsf{R}^n, \mathsf{R})$-knot, at least one $\pi_i \circ f$ $(i = 1, \dots, n)$ is cofinal; hence, we have $2^n - 1$ possible directions. Each of these directions D can be realized by the $(\mathsf{R}^n, \mathsf{R})$-knot f_D given by $f_D(x) = x \cdot D^T$. Now, we just need to prove that an $(\mathsf{R}^n, \mathsf{R})$-knot f with direction D is equivalent to f_D.

By a permutation of the indices, it may be assumed that $\pi_i \circ f$ is bounded for $i = 1, \dots, s$ and cofinal for $i = s+1, \dots, n$. Consider the covers $\{[x_\alpha, x_{\alpha+1}]\}_{\alpha<\omega_1}$ of R and $\{D_\alpha\}_{0<\alpha<\omega_1}$

of $f([x_1, \omega_1[)$ given by Lemma 2.4. For $i = 1, \dots, s$, take an isotopy ϕ_t^i of R' between $\pi_i \circ f(x_1) = c_i$ and 0; the isotopy $\phi_t = (\phi_t^1, \dots, \phi_t^s, \mathrm{id}, \dots, \mathrm{id})$ then sends $f([x_1, \omega_1[)$ on $\{0\} \times \mathsf{R}^{n-s}$. Using Proposition 2.5 d), we may assume that $f = f_D$ on $[0, x_1]$. By Proposition 2.5 b), we have an isotopy $(\phi_\alpha)_t$ in D_α between $f|_{[x_\alpha, x_{\alpha+1}]}$ and $f_D|_{[x_\alpha, x_{\alpha+1}]}$ for all $0 < \alpha < \omega_1$. Extending this isotopy by the identity outside $\cup_\alpha D_\alpha$, it follows that $f \sim f_D$. $\square$

Theorem 3.7. *There are exactly* $(3^n - 1)^2 + (n-2)2^{n-1}$ *non-equivalent differentiable* $(\mathsf{L}^n, \mathsf{L})$*-knots if* $n \geqq 4$.

Proof. We have $(3^n)^2$ possible directions for a continuous map $f : \mathsf{L} \to \mathsf{L}^n$. If f is an embedding, $\delta(f|_{\mathsf{L}_-})$ and $\delta(f|_{\mathsf{L}_+})$ are non-zero, giving $(3^n - 1)^2$ directions. Furthermore, a direction matrix with identical columns and no zero coefficient cannot be realized by an embedding (the argument is similar to the case $n = 2$). Therefore, we are left with $(3^n - 1)^2 - 2^n$ directions, which can all clearly be realized by $(\mathsf{L}^n, \mathsf{L})$-knots. Finally, to any $(\mathsf{L}^n, \mathsf{L})$-direction corresponds exactly one class of knots, except if it is a double direction; in this case, there are exactly two classes of knots with this direction. We thus have $(3^n - 1)^2 - 2^n + n2^{n-1} = (3^n - 1)^2 + (n-2)2^{n-1}$ different classes of knots. $\square$

Differentiable $(\mathsf{R}^3, \mathsf{R})$ and $(\mathsf{L}^3, \mathsf{L})$-knots.

Theorem 3.8. *There are* $2^3 - 1 = 7$ *non-equivalent differentiable* $(\mathsf{R}^3, \mathsf{R})$*-knots.*

We shall need these two lemmas:

Lemma 3.9. *Let f be a differentiable* $(\mathsf{R}^3, \mathsf{R})$*-knot such that $\pi_i \circ f$ is cofinal for $i = 1, 2, 3$, and let* $\mathcal{P}^3 \stackrel{\text{def}}{=} \{[x_\alpha, x_{\alpha+1}]^3\}_{0<\alpha<\omega_1}$ *with x_α as in Lemma 2.4. Then, for all but finitely many α, there is an index i with $\pi_i \circ f$ monotone on $[x_\alpha, x_{\alpha+1}]$.*

Lemma 3.10. *Let $f : [0, 1] \to [0, 1]^3$ be a differentiable embedding, and let B be a compact 3-ball such that $B \supset [0, 1]^3$ and $\partial B \cap [0, 1]^3 = \{(0, 0, 0), (1, 1, 1)\}$. Suppose that $f(0) = (0, 0, 0)$, $f(1) = (1, 1, 1)$, and that $\pi_i \circ f$ is monotone for at least one i; then, there is an isotopy ϕ_t of B, which is the identity on ∂B, such that $\phi_0 = \mathrm{id}$ and $\phi_1 \circ f(x) = (x, x, x)$.*

Proof of Lemma 3.9. Otherwise, let $\{\beta_m\}_{m<\omega}$ be an increasing sequence of ordinals such that $\pi_i \circ f$ is not monotone on $I_{\beta_m} = [x_{\beta_m}, x_{\beta_m+1}]$ for $i = 1, 2, 3$, and let us denote by β the limit of this sequence. For $m < \omega$ and $i = 1, 2, 3$, choose $x_{i,m}, y_{i,m} \in I_{\beta_m}$ such that $\pi_i \circ f$ is decreasing in a neighborhood of $x_{i,m}$ and increasing in a neighborhood of $y_{i,m}$. In any chart containing $[0, x_\beta]$, $(\pi_i \circ f)'(x_{i,m}) \leqq 0$ and $(\pi_i \circ f)'(y_{i,m}) \geqq 0$ for any $m < \omega$ and $i = 1, 2, 3$. By construction, $\lim_{m\to\infty} x_{i,m} = \lim_{m\to\infty} y_{j,m} \stackrel{\text{def}}{=} u$ for any i, j. By continuity, $(\pi_i \circ f)'(u) = 0$ for $i = 1, 2, 3$, contradicting the 1-regularity of f. $\square$

Proof of Lemma 3.10. Apply a descending curve argument. $\square$

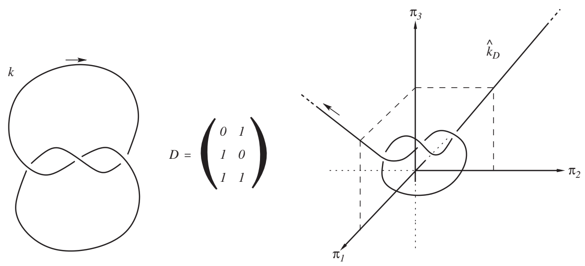

Figure 4. The trefoil knot and its D-associate.

Proof of Theorem 3.8. Clearly, there are 7 possible $(\mathsf{R}^3, \mathsf{R})$-directions. Let us prove that two knots f, g with $D(f) = D(g)$ are equivalent.

First, consider the case where $\pi_i \circ f$ is bounded for some index i (let us say that $i = 1$). By Lemma 2.2 a), there is an x_1 in R such that $\pi_1 \circ f(x) = c$ for all $x \geqq x_1$. Let D_0 be a compact 3-ball in $(\mathsf{R}')^3$ such that $f([0, x_1]) \subset D_0$, $f^{-1}(\partial D_0) = \{x_1\}$ and $f([0, x_1]) \cap D_0 = \{f(x_1)\}$. By Proposition 2.5 d), f is isotopic to an $(\mathsf{R}^2, \mathsf{R})$-knot in $\{c\} \times \mathsf{R}^2$. We can conclude with Theorem 3.3.

Now, let us assume that $\pi_i \circ f$ is cofinal for $i = 1, 2, 3$. By Lemmas 3.9 and 3.10, there is an x in R such that $f|_{[x,\omega_1[}$ is isotopic (in $[x, \omega_1[^3$) to the knot f_D given by $f_D(x) = (x, x, x)$. We then proceed as before for $f|_{[0,x]}$. □

Let us now turn to $(\mathsf{L}^3, \mathsf{L})$-knots. We will assume that $\mathbb{S}^3$ and L^3 have a fixed orientation. Let k be an oriented differentiable $(\mathbb{S}^3, \mathbb{S}^1)$-knot, and let D be an $(\mathsf{L}^3, \mathsf{L})$-direction. If D is not a double direction, there is clearly a unique equivalence class of 'unknotted' differentiable $(\mathsf{L}^3, \mathsf{L})$-knot with direction D; as before, let us denote a representant by f_D. If D is a double direction, there are exactly two such 'unknotted' $(\mathsf{L}^3, \mathsf{L})$-knots with direction D; to simplify the notation we shall abusively denote both by f_D.

Definition 3.11. A differentiable $(\mathsf{L}^3, \mathsf{L})$-knot $\widehat{k}_D$ equivalent to the oriented connected sum $f_D \# k$ is called a D-associate of k. (See Figure 4 for an example.)

It is easy to show that, up to the subtleties due to the double directions, the equivalence class of $\widehat{k}_D$ only depends on the equivalence class of the oriented $(\mathbb{S}^3, \mathbb{S}^1)$-knot k and on the direction D: just follow the classical proof that the connected sum of oriented differentiable $(\mathbb{S}^3, \mathbb{S}^1)$-knots is well defined. Counting the directions as in Theorem 3.7, we find immediately:

Proposition 3.12. *A differentiable* $(\mathbb{S}^3, \mathbb{S}^1)$*-knot has exactly* 680 *non-equivalent associates.*

Moreover, the classification of smooth $(\mathsf{L}^3, \mathsf{L})$-knots with direction D is equivalent to the classification of differentiable oriented $(\mathbb{S}^3, \mathbb{S}^1)$-knot. In other words:

Theorem 3.13. *A differentiable* $(\mathsf{L}^3, \mathsf{L})$*-knot is the associate of a unique type of differentiable oriented* $(\mathbb{S}^3, \mathbb{S}^1)$*-knot.*

Proof. Let f be a differentiable $(\mathsf{L}^3, \mathsf{L})$-knot with direction D. Using the same argument as in Theorem 3.8, we see that for some $x_1 \in \mathsf{L}_+$, there is an isotopy φ_t of $X = \mathsf{L}^3 \backslash]-x_1, x_1[^3$, keeping ∂X fixed, such that $\varphi_0 = \mathrm{id}_X$ and $\varphi_1 \circ f|_{f^{-1}(X)} = f_D|_{f_D^{-1}(X)}$. Let k be the smooth oriented $(\mathbb{S}^3, \mathbb{S}^1)$-knot obtained by attaching both ends of $f|_{f^{-1}([-x_1,x_1]^3)}$ with an unknotted arc in X. By construction, f is equivalent to $f_D \# k$, so f is the D-associate of k.

Now, let k, k' be two differentiable $(\mathbb{S}^3, \mathbb{S}^1)$-knots such that there is an ambient isotopy ϕ_t of L^3 with $\phi_1 \circ \widehat{k}_D = \widehat{k}'_D$. If U_α denotes the open cube $]-\alpha, \alpha[^3$ in L^3, then $\{U_\alpha \mid \alpha < \omega_1\}$ is a *canonical sequence* in the sense of [2, p. 147]. By the proposition on the same page, the set

$$F_\phi \stackrel{\text{def}}{=} \{\alpha < \omega_1 \mid \phi_t(\overline{U}_\alpha \backslash U_\alpha) = \overline{U}_\alpha \backslash U_\alpha \ \forall t\}$$

is cofinal in $W(\omega_1)$. (The proposition is stated for ω-bounded 2-surfaces, but its proof shows that it also holds for R^n and L^n.) Take α large enough such that $\widehat{k}_D|_{\mathsf{L}^3 \backslash U_\alpha}$ and $\widehat{k}'_D|_{\mathsf{L}^3 \backslash U_\alpha}$ are equal to $f_D|_{\mathsf{L}^3 \backslash U_\alpha}$; let us denote by $\{P_1, P_2\}$ the two points of $\mathrm{Im}\, \widehat{k}_D \cap (\overline{U}_\alpha \backslash U_\alpha) = \mathrm{Im}\, \widehat{k}'_D \cap (\overline{U}_\alpha \backslash U_\alpha)$. For all t, $\phi_t| : \overline{U}_\alpha \backslash U_\alpha \to \overline{U}_\alpha \backslash U_\alpha$ is a homeomorphism isotopic (rel $\{P_1, P_2\}$) to the identity. Therefore, $\phi_t|$ can be extended to a homeomorphism $\widetilde{\phi}_t : B \to B$, where B is a compact ball containing $\overline{U}_\alpha$, with $\widetilde{\phi}_t$ keeping ∂B fixed. Extending $\widetilde{\phi}_t$ with the identity on $\mathbb{S}^3 \backslash B$, we get an isotopy between k and k'. $\square$

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Received: 1 October 2002; revised manuscript accepted: 10 December 2003

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